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THE EFFECTIVE ELASTIC MODULI OF MICROCRACKED COMPOSITE MATERIALS

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Abstract—The self-consistent mechanics method has been widely used to estimate the macroscopic elastic moduli of solids containing microvoids and inclusions. Another method based on the crack energy release and potential energy balance has also been used to estimate the overall elastic moduli of a microcracked solid. It is shown here that these two approaches are equivalent for microcracked solids, thus one can take advantage of both methods to estimate the elastic moduli of inclusion-crack-matrix composites, i.e. microcracked composite material. A solid containing spherical inclusions and randomly distributed penny-shaped cracks is then studied. The effective elastic moduli of a solid with spherical inclusions and parallel-distributed penny-shaped cracks are also studied. It is established that the effects of inclusions and microcracks on overall moduli are approximately decoupled for stiff inclusions, which are in most metal matrix composites. This conclusion is particularly useful since one may then obtain the moduli of composites by a simple two-step estimation. For compliant inclusions, including the limiting case of voids, the decoupling does not hold.

1. INTRODUCTION

The study of materials containing cracks, voids, and other inclusions has diverse applications in several fields. For instance, ceramic components and intermetallics bear various defects and inclusions either through fabrication or modifying processes. Also, voids and cracks may develop during the service period of a structure. The effective moduli of materials containing microdefects offer important insights into problems involving engineering structures.

Two approaches to investigate a solid containing various defects and inclusions can be identified. The micro-scale approach focuses on the individual inclusions, defects, and local stresses around them. This approach details the evolution of local defects and rigorously accounts for interactions among inclusions and defects (Horii and Nemat-Nasser, 1985; Kachanov, 1985; Hu and Chandra, 1992b). The macro-scale approach attempts to estimate the effect of microdefects on the overall macroscopic material properties by averaging over defect distributions. The goal of the latter approach is to estimate the macroscopic properties of composite materials in terms of properties and volume concentrations of individual phases. It includes the following approximate techniques: the self-consistent method (Budiansky, 1965; Hill, 1965; Budiansky and O'Connell, 1976; Ju, 1991), differential method (Henyey and Pomphery, 1982; Norris, 1985; Hashin, 1988; Bassani, 1991), three-phase composite model (Smith, 1974; Christensen and Lo, 1979), and Mori-Tanaka's theory (Taya and Chou, 1981; Taya and Mura, 1981; Weng, 1984; Benveniste, 1987).

The effective properties of cracked solids have received rekindled attention in recent years due to their application to damage mechanics [see Kachanov (1992) for an extensive literature review]. The limited work related to microcracked composites includes an analysis of a short fiber-reinforced composite containing fiber-end cracks (Taya and Mura, 1981). Bassani (1991) studied a solid containing voids and/or cracks using the differential method and assuming a special filling path, i.e. a particular correlation between void volume fraction and crack density. As discussed by Norris (1985), effective modulus dependence on a filling path imposes serious questions because one filling path can yield effective properties that are significantly different from another.

The work by Budiansky and O'Connell (1976) for a solid with randomly distributed cracks is important to the present study in several aspects. First, they implemented the self-consistent scheme to approximately take into consideration the interactions among cracks of large concentration. This is an extension of the self-consistent mechanics for composite materials developed independently by Budiansky (1965) and Hill (1965) and discussed later in this paper. Second, Budiansky and O'Connell emphasize that the effective elastic moduli of a microcracked solid vary with a crack density parameter ε rather than with crack porosity. Practically speaking, this means that the volume concentration of pore space may not be a useful measure of the effects of cracks on moduli. A critical value of the crack density, $\varepsilon = 9/16$, is identified at which the elastic moduli of the cracked composite vanish. Finally, and most importantly in terms of the energy balance consideration, crack-induced potential energy change entered their formulation with explicit physical interpretation, but was not at all explicit in the earlier self-consistent mechanics methods (Budiansky, 1965; Hill, 1965). Budiansky and O'Connell's (1976) crack energy release approach is particularly useful for the estimation of overall macroscopic properties of a microcracked solid. Their analysis has been generalized to various crack distributions that lead to anisotropic stress–strain behavior [transversely isotropic (Hoenig, 1979; Hu and Huang, 1991) and general anisotropic (Huang *et al.*, 1992)].

Previous studies on inclusions and cracks using self-consistent mechanics have been quite separate. Studies on inclusions in matrix material are based on Budiansky's (1965) and Hill's (1965) work, while Budiansky and O'Connell's work (1976) forms the basis for research on microcracked solids. The approaches to these two categories of problems look very different. Motivated by the aforementioned observations, this paper presents a study of the effective moduli for a solid containing inclusions and cracks within the framework of self-consistent mechanics. The matrix material and inclusions are assumed isotropic, with different elastic moduli and Poisson's ratios. The macroscopic behavior of a composite can be isotropic or anisotropic, depending on the microcrack distributions.

In Section 2, the crack energy release approach presented by Budiansky and O'Connell (1976) is found to be equivalent to a limiting case of the self-consistent mechanics developed by Budiansky (1965) and Hill (1965) in the sense that a crack is considered as the limit of a void if one of its dimensions approaches zero. This equivalence ensures that for a crack-inclusion-matrix three-phase composite, one can take advantage of both approaches to set up the governing equations and to evaluate the effective elastic moduli of a solid with inclusions and microcracks embedded, i.e. microcracked composite material. Section 3 addresses the effective moduli of a composite containing randomly distributed penny-shaped cracks and spherical inclusions. The microcracked composite is isotropic. Section 4 presents the effective moduli of a solid containing spherical inclusions and parallel distributions of penny-shaped cracks. The microcracked composite is not isotropic, but is transversely isotropic.

2. SELF-CONSISTENT MECHANICS FOR COMPOSITE MATERIALS AND CRACKED SOLIDS

Budiansky (1965) and Hill (1965) independently developed a self-consistent mechanics method to estimate the effective moduli of composite materials. Their theory forms the basis for the present work. A summary applying their theory for general anisotropic composite materials is presented in this section.

Consider a large cube of multiphase composite material composed of a coherent mixture of several isotropic elastic materials. The spatial distributions of the phases are assumed to be such that, generally, the composite material is homogeneous. Let V denote the total volume of the composite material, N the total number of phases, and c_I ($I = 1, 2, \ldots, N, \Sigma_{I=1}^{N} c_I = 1$) the volume concentration of the Ith phase. Thus, $V_I = c_I V$ is the volume of the Ith phase. The shear modulus, G_I, and bulk modulus, K_I, of the Ith phase are related to the Young's modulus, E_I, and Poisson's ratio, ν_I, by $G_I = E_I/[2(1+\nu_I)]$ and $K_I = E_I/[3(1-2\nu_I)]$. In general, the composite material is anisotropic, due to the geometrical shapes and spatial orientations of inclusions, and is characterized by the following general stress–strain relation:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \varepsilon_{ij} = C_{ijkl}^{-1}\sigma_{kl}, \tag{1}$$

where C_{ijkl} is the elastic moduli tensor of the composite material. To determine C_{ijkl}, or its inverse C^{-1}_{ijkl} (tensor of elastic constants), apply a uniform remote stress $\sigma_{ij} = \sigma^0_{ij}$ to the surface of the cube of composite material. The corresponding remote strain is $\varepsilon^0_{ij} = C^{-1}_{ijkl}\sigma^0_{kl}$. The strain energy of the composite material is given exactly by (Hill, 1963)

$$U = \tfrac{1}{2}\sigma^0_{ij}\varepsilon^0_{ij}V = \tfrac{1}{2}C^{-1}_{ijkl}\sigma^0_{ij}\sigma^0_{kl}V. \tag{2}$$

Also, in terms of the individual stress σ_{ij}, strain ε_{ij}, modulus E_I, and Poisson's ratio ν_I of the various phases (Budiansky, 1965),

$$\begin{aligned} U &= \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = \tfrac{1}{2}\sigma^0_{ij}\int_V \varepsilon_{ij}\,\mathrm{d}V \\ &= \tfrac{1}{2}\sigma^0_{ij}\int_V\left(\frac{1+\nu_N}{E_N}\sigma_{ij} - \frac{\nu_N}{E_N}\sigma_{kk}\delta_{ij}\right)\mathrm{d}V + \tfrac{1}{2}\sigma^0_{ij}\int_V\left(\varepsilon_{ij} - \frac{1+\nu_N}{E_N}\sigma_{ij} + \frac{\nu_N}{E_N}\sigma_{kk}\delta_{ij}\right)\mathrm{d}V \\ &= \frac{1}{2}\left(\frac{1+\nu_N}{E_N}\sigma^0_{ij}\sigma^0_{ij} - \frac{\nu_N}{E_N}\sigma^0_{kk}\sigma^0_{ll}\right)V + \tfrac{1}{2}\sigma^0_{ij}\sum_{I=1}^{N-1}\int_{V_I}\left[\left(1 - \frac{1+\nu_N}{1+\nu_I}\frac{E_I}{E_N}\right)\varepsilon_{ij}\right. \\ &\quad \left. + \frac{\nu_N - \nu_I}{(1-2\nu_I)(1+\nu_I)}\frac{E_I}{E_N}\varepsilon_{kk}\delta_{ij}\right]\mathrm{d}V \\ &= \frac{1}{2}\left\{\frac{1+\nu_N}{E_N}\sigma^0_{ij}\sigma^0_{ij} - \frac{\nu_N}{E_N}\sigma^0_{kk}\sigma^0_{ll} + \sum_{I=1}^{N-1} c_I\left[\left(1 - \frac{1+\nu_N}{1+\nu_I}\frac{E_I}{E_N}\right)\sigma^0_{ij}\bar{\varepsilon}_{ij}\right.\right. \\ &\quad \left.\left. + \frac{\nu_N - \nu_I}{(1-2\nu_I)(1+\nu_I)}\frac{E_I}{E_N}\sigma^0_{kk}\bar{\varepsilon}_{ll}\right]\right\}V, \end{aligned} \tag{3}$$

where $\bar{\varepsilon}_{ij} = 1/V_I\int_{V_I}\varepsilon_{ij}\,\mathrm{d}V$ is the average value of ε_{ij} in the Ith phase. Comparison of eqns (2) and (3) leads to

$$\begin{aligned} C^{-1}_{ijkl}\sigma^0_{ij}\sigma^0_{kl} &= \frac{1+\nu_N}{E_N}\sigma^0_{ij}\sigma^0_{ij} - \frac{\nu_N}{E_N}\sigma^0_{kk}\sigma^0_{ll} \\ &\quad + \sum_{I=1}^{N-1} c_I\left[\left(1 - \frac{1+\nu_N}{1+\nu_I}\frac{E_I}{E_N}\right)\sigma^0_{ij}\bar{\varepsilon}_{ij} + \frac{\nu_N-\nu_I}{(1-2\nu_I)(1+\nu_I)}\frac{E_I}{E_N}\sigma^0_{kk}\bar{\varepsilon}_{ll}\right]. \end{aligned} \tag{4}$$

Now, the self-consistent approximation (Budiansky, 1965; Hill, 1965) is used such that ε_{ij} is approximated by the strain that would occur in an isolated inclusion of the Ith material embedded in an infinite elastic matrix subjected to remote stress $\sigma_{ij} = \sigma^0_{ij}$ at infinity and having the as-yet-unknown elastic constants C^{-1}_{ijkl} of the composite materials. An exact solution of $\bar{\varepsilon}_{ij}$ is given by Eshelby (1957) for an ellipsoidal inclusion embedded in an anisotropic matrix subject to uniform stress at infinity. Solutions for specific inclusion geometries and elastic properties of inclusions and matrices can be found in Mura (1982). Equation (4) is the basic equation to determine the elastic constants, C^{-1}_{ijkl}, of the composite material, and it leads to a group of independent equations for C^{-1}_{ijkl} with different loadings, $\sigma_{ij}{}^0$, applied. It should be pointed out that the elastic constants, C^{-1}_{ijkl}, of the composite depend on the elastic moduli E_I, Poisson's ratio ν_I, and volume concentration c_I of the individual phase ($I = 1, 2, \ldots, N$) and possibly on the shape and orientation of inclusions, but not on the remote stresses σ^0_{ij}, although σ^0_{ij} show up in eqn (4). If the composite material is isotropic, eqn (4) gives two independent equations for C^{-1}_{ijkl}, which are identical to those given by Budiansky (1965). Equation (4) leads to 5 and 21 equations for C^{-1}_{ijkl} for a transversely isotropic solid and for a general anisotropic solid, respectively.

Budiansky and O'Connell (1976) present a self-consistent estimate of the effective elastic moduli of a microcracked solid by a different approach. They do not start with eqn (4), but utilize the potential energy balance and the relationship between potential energy change and the crack energy release rate. Their approach is particularly useful for solids with microcracks embedded since they avoid the difficulty of evaluating inclusion strain $\bar{\varepsilon}_{ij}$ for microcracks in eqn (4). Penny-shaped cracks were assumed to be randomly distributed in the matrix material such that the crack material behaves like an isotropic solid. The moduli of the cracked solid were found, depending not on the porosity volume concentration, but on a newly introduced parameter, the crack density,

$$\varepsilon = N\langle a^3\rangle, \tag{5}$$

where N is the number of cracks per unit volume, a is the radius of the penny-shaped crack, and $\langle\cdot\rangle$ is the average of the argument. A critical value of crack density, $\varepsilon = 9/16$, was established at which the effective elastic moduli of the cracked solid vanish.

Although Budiansky and O'Connell's (1976) analysis of a cracked solid was based on the potential energy balance, it is shown in the following that their results still fall into the general framework of self-consistent mechanics of composite materials (Budiansky, 1965; Hill, 1965). A penny-shaped crack can be considered as the limit of an oblate spheroidal cavity with $a_1 = a_2 = a$ and $a_3 \to 0$, where a_1, a_2 and a_3 are the half-axes of the spheroid. The basic equation [eqn (4)], for a solid with spheroidal cavities embedded, becomes

$$\frac{1+\bar{\nu}}{\bar{E}}\sigma_{ij}^0\sigma_{ij}^0 - \frac{\bar{\nu}}{\bar{E}}\sigma_{kk}^0\sigma_{ll}^0 = \frac{1+\nu_N}{E_N}\sigma_{ij}^0\sigma_{ij}^0 - \frac{\nu_N}{E_N}\sigma_{kk}^0\sigma_{ll}^0 + c\sigma_{ij}^0\bar{\varepsilon}_{ij}, \tag{6}$$

where $\bar{E}$ and E_N are the Young's modulus and $\bar{\nu}$ and ν_N are Poisson's ratio of the cracked solid and matrix material, respectively; c is the volume concentration of the cavity. Note that

$$c\bar{\varepsilon}_{ij} = \frac{1}{V}\int_{V_{\text{cavity}}} \varepsilon_{ij}\,\mathrm{d}V = \frac{1}{V}\sum_{\text{all cavities}} \varepsilon_{ij}\cdot\tfrac{4}{3}\pi a^2 a_3, \tag{7}$$

where V_{cavity} is the total volume of the cavities, V is the total volume of the solid, $4/3(\pi a^2 a_3)$ is the volume of each cavity, and ε_{ij}, which is uniform within each cavity, was given by Eshelby (1957). In the case of remote hydrostatic tension, $\sigma_{ij}^0 = \sigma^0\delta_{ij}$, the strain, ε_{ij}, within the cavity has the asymptotic form (Eshelby, 1957; Mura, 1982)

$$\varepsilon_{33} = \frac{4}{\pi}\frac{1-\bar{\nu}^2}{\bar{E}}\frac{a}{a_3}\sigma_0 + O(1), \quad \text{others} = O(1). \tag{8}$$

Thus, eqn (7), in the limit of cracks $(a_3/a \to 0)$, gives

$$c\bar{\varepsilon}_{kk} = \frac{1}{V}\Sigma a^3\cdot\frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\sigma_0 = N\langle a^3\rangle\frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\sigma_0. \tag{9}$$

Substituting into the basic equation [eqn (4)], one finds

$$\frac{3(1-2\bar{\nu})}{\bar{E}} = \frac{3(1-2\nu_N)}{E_N} + \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\varepsilon, \tag{10}$$

which is exactly one of the governing equations for determination of effective moduli given by Budiansky and O'Connell (1976). If other kinds of the remote loading σ_{ij}^0 are applied, one can similarly derive other governing equations of effective moduli identical to Budiansky and O'Connell's. This shows that their analysis of a cracked solid is consistent with the self-consistent mechanics of composite materials (Budiansky, 1965; Hill, 1965). One can therefore use the basic equation [eqn (4)], to calculate the strain energy for regular inclusions while evaluating the strain energy change due to cracks following the approach proposed by Budiansky and O'Connell. This combined approach is adopted in the following sections.

3. EFFECTIVE MODULI OF A COMPOSITE MATERIAL WITH INCLUSIONS AND RANDOMLY DISTRIBUTED CRACKS

The elastic moduli of a solid with spherical inclusions and penny-shaped microcracks embedded are estimated in this section. The interactions among matrix material, inclusions, and cracks are accounted for implicitly by the self-consistent method. The microcracks and inclusions are randomly distributed in size, orientation and location, as shown in Fig. 1. Thus, the composite material behaves like an isotropic solid, with Young's modulus, $\bar{E}$, and Poisson's ratio, $\bar{v}$, to be determined. There are three phases in this composite: Matrix material with Young's modulus E, Poisson's ratio v, and volume concentration $1-c$; spherical inclusions with Young's modulus E_I, Poisson's ratio v_I, and volume concentration c; and penny-shaped cracks with the crack density $\varepsilon = N\langle a^3\rangle$, as defined in eqn (5).

For remote hydrostatic tension $\sigma_{ij}^0 = \sigma^0\delta_{ij}$, the strain $\bar{\varepsilon}_{ij}$ in the inclusion is $\bar{\varepsilon}_{ij} = 1/3\sigma^0\delta_{ij}/\{\bar{K}+(1+\bar{v})/[3(1-\bar{v})](K_I-\bar{K})\}$ (Eshelby, 1957), where $\bar{K}=\bar{E}/(3-6\bar{v})$ and $K_I = E_I/(3-6v_I)$ are the bulk moduli of the composite and inclusions, respectively. The term associated with cracks in eqn (4) can be calculated by Budiansky and O'Connell's (1976) crack energy release approach, as discussed in Section 2, which gives $16(1-\bar{v}^2)\varepsilon\sigma^{0^2}/3\bar{E}$, where ε is the crack density. Equation (4) gives

$$\frac{\bar{K}}{K}+c\left(1-\frac{K_I}{K}\right)\frac{\bar{K}}{\bar{K}+\dfrac{1+\bar{v}}{3(1-\bar{v})}(K_I-\bar{K})}+\frac{16}{9}\frac{1-\bar{v}^2}{1-2\bar{v}}\varepsilon = 1, \tag{11}$$

where $K = E/(3-6v)$ is the bulk modulus of the matrix material. In the limit $\varepsilon = 0$, i.e. no cracks, eqn (11) is consistent with Budiansky's (1965) and Hill's (1965) work on the estimate of moduli of multi-phase composite materials. In the other limit, $c = 0$, i.e. no inclusions, eqn (11) is identical to Budiansky and O'Connell's (1976) results for a cracked solid.

For the remote pure shear ($\sigma_{23}^0 \neq 0$, others $= 0$), the basic equation [eqn (4)], after the proper averaging over crack orientations, gives

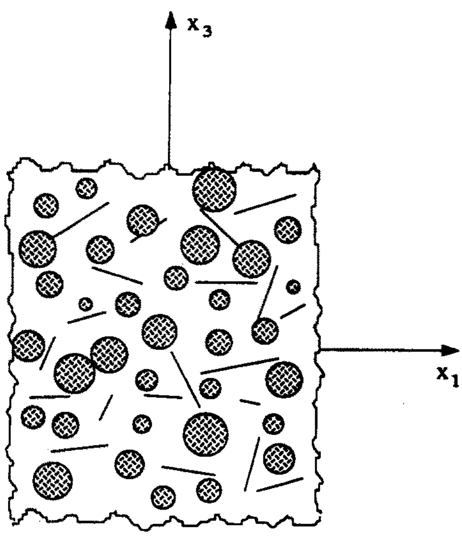

Fig. 1. Schematic diagram for randomly distributed cracks and inclusions.

$$\frac{\bar{G}}{G}+c\left(1-\frac{G_I}{G}\right)\frac{\bar{G}}{\bar{G}+\frac{8-10\bar{\nu}}{15(1-\bar{\nu})}(G_I-\bar{G})}+\frac{32}{45}\frac{(1-\bar{\nu})(5-\bar{\nu})}{2-\bar{\nu}}\varepsilon = 1, \tag{12}$$

where $\bar{G} = \bar{E}/[2(1+\bar{\nu})]$, $G = E/[2(1+\nu)]$, and $G_I = E_I/[2(1+\nu_I)]$ are the shear moduli of the composite, matrix and inclusions, respectively. Equations (11) and (12) are the governing equations to determine the Young's modulus of microcracked composites, $\bar{E}$, and Poisson's ratio, $\bar{\nu}$. The application of remote loading, σ_{ij}^0, other than hydrostatic tension or pure shear, leads to equations that are not independent of eqns (11) and (12).

For the limiting case of spherical voids ($E_I = G_I = K_I = 0$) with randomly distributed cracks, eqns (11) and (12) are reduced to

$$\frac{\bar{K}}{K}+\frac{3(1-\bar{\nu})}{2(1-2\bar{\nu})}c+\frac{16}{9}\frac{1-\bar{\nu}^2}{1-2\bar{\nu}}\varepsilon = 1 \tag{13a}$$

and

$$\frac{\bar{G}}{G}+\frac{15(1-\bar{\nu})}{7-5\bar{\nu}}c+\frac{32}{45}\frac{(1-\bar{\nu})(5-\bar{\nu})}{2-\bar{\nu}}\varepsilon = 1. \tag{13b}$$

At the other extreme, rigid inclusions ($E_I = G_I = K_I = \infty$) and cracks, eqns (11) and (12) become

$$\frac{\bar{K}}{K}\left[1-\frac{3(1-\bar{\nu})}{1+\bar{\nu}}c\right]+\frac{16}{9}\frac{1-\bar{\nu}^2}{1-2\bar{\nu}}\varepsilon = 1 \tag{14a}$$

and

$$\frac{\bar{G}}{G}\left[1-\frac{15(1-\bar{\nu})}{8-10\bar{\nu}}c\right]+\frac{32}{45}\frac{(1-\bar{\nu})(5-\bar{\nu})}{2-\bar{\nu}}\varepsilon = 1. \tag{14b}$$

It can be established from eqns (11) and (12) that the elastic moduli $\bar{E}$, $\bar{G}$ and $\bar{K}$ of the composite vanish as the crack density ε reaches 9/16, independent of the inclusion volume concentration c. This condition, $\varepsilon = 9/16$ for vanishing moduli, is identical to that established by Budiansky and O'Connell (1976) for a cracked matrix material without inclusions. The moduli of composite materials depend strongly on the number of inclusions and cracks, but the moment at which moduli of the composite materials vanish depends only on the attainment of a critical value of the crack density,

$$\varepsilon = \tfrac{9}{16}. \tag{15}$$

The criterion [eqn (15)] also holds for a composite material with rigid inclusions and cracks, plus a constraint on the volume concentration of inclusions, $c < 1/3$, as determined by eqn (14). However, the criterion for vanishing moduli of a composite material with voids and cracks is different because the void volume concentration c plays a role. The following approximate criterion for vanishing moduli of a composite material with voids and cracks, within 0.4% error, is obtained from eqn (13):

$$\tfrac{16}{9}\varepsilon+2c = 1. \tag{16}$$

Equation (16) is reduced to $c = 1/2$ if there is no crack ($\varepsilon = 0$), which is consistent with Budiansky (1965).

The numerical solution of eqns (11) and (12) shows that the Poisson's ratio, $\bar{\nu}$, of the composite material depends very weakly on the inclusion volume concentration, c, and the

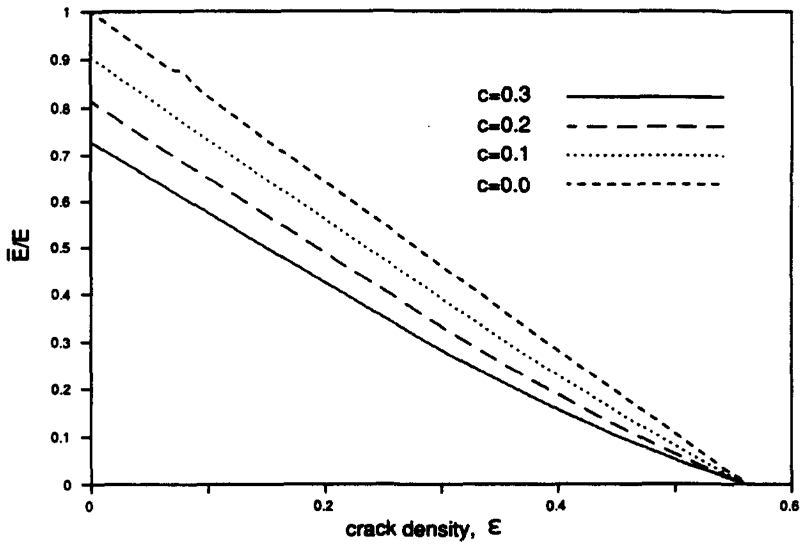

Fig. 2. Variation of $\bar{E}/E$ with crack density ε: Randomly distributed cracks and inclusions $E_I/E = 1/3$.

ratio of the inclusion and matrix Young's modulus, E_I/E. Thus, $\bar{\nu}$ depends approximately only on the matrix Poisson's ratio, ν, and the crack density, ε, and is well approximated by

$$\bar{\nu} = \nu(1 - \tfrac{16}{9}\varepsilon). \tag{17}$$

The normalized Young's modulus, $\bar{E}/E$, versus crack density, ε, is presented for various inclusion volume concentrations, c, in Figs 2 and 3, for compliant inclusions, $E_I/E = 1/3$, and stiff inclusions, $E_I/E = 3$, respectively. The Poisson's ratios of the matrix and inclusions are taken as 0.25 and 0.33, respectively. For stiff inclusions ($E_I/E > 1$), which is the case for most metal matrix composites and some ceramic matrix composites, the Young's modulus of the composite, $\bar{E}$, versus the crack density, ε, is approximately linear, as shown in Fig. 3. This gives

$$\bar{E} \cong \bar{E}(\varepsilon = 0) \cdot (1 - \tfrac{16}{9}\varepsilon), \tag{18}$$

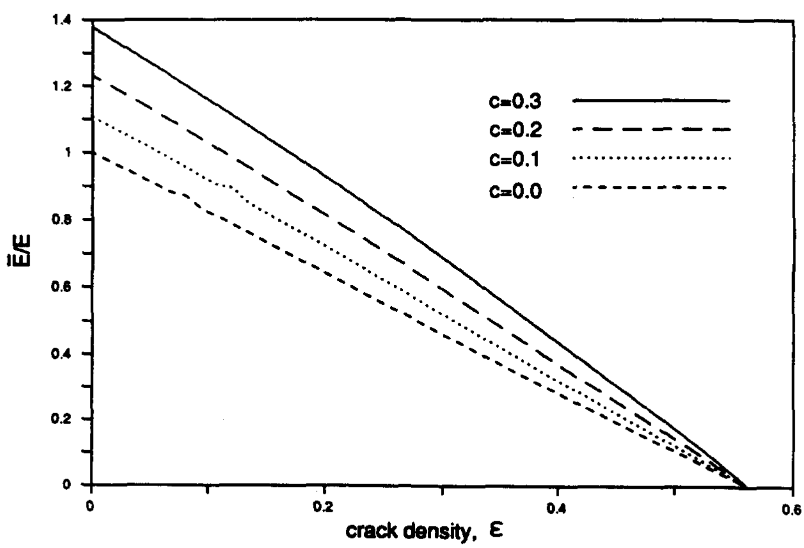

Fig. 3. Variation of $\bar{E}/E$ with crack density ε: Randomly distributed cracks and inclusions $E_I/E = 3$.

where $\bar{E}(\varepsilon = 0)$ is the Young's modulus of the uncracked composites. The effect of microcracks and stiff inclusions is approximately decoupled. One can include the effect of microcracking by multiplying the factor $(1 - 16/9\varepsilon)$ by the uncracked composite's modulus, $\bar{E}(\varepsilon = 0)$, which was given by Budiansky (1965) and Hill (1965).

For compliant inclusions ($E_I/E < 1$), as the inclusion volume concentration, c, increases, the Young's modulus, $\bar{E}$, versus ε deviates more and more from a linear relation (Fig. 2). Thus, eqn (18) does not hold for compliant inclusions, including the limiting case of microvoids, and one must start from eqns (11) and (12) in order to evaluate the Young's modulus of the composite.

4. EFFECTIVE MODULI OF A COMPOSITE MATERIAL WITH INCLUSIONS AND NONRANDOMLY DISTRIBUTED CRACKS

Nonrandomly distributed cracks are frequently discovered in engineered materials. For example, a layer of cracks is introduced in ceramics to reduce the modulus of the hard material in order to make the grinding of the hard material easier (Hu and Chandra, 1992a). Hu and Huang (1991) and Huang *et al.* (1992) studied the effect of distributed, long, tunnel cracks on the macroscopic elastic properties of materials. The elastic moduli of a composite material with parallel penny-shaped cracks are studied in this section. The spherical inclusions are embedded in the matrix. As in the previous section, there are three phases in this composite: matrix material, inclusions, and parallel penny-shaped cracks with the crack density $\varepsilon = N\langle a^3 \rangle$, as defined in eqn (5). These penny-shaped cracks are assumed normal to the x_3-axis (parallel to the x_1–x_2 plane, Fig. 1).

The cracked composite is not isotropic due to parallel cracks. The tensile modulus in the direction normal to the crack plane is reduced more than in the other direction. The cracked composite has the following transversely isotropic stress–strain relation, with the principal axis x_3 normal to the crack planes:

$$\left.\begin{aligned} \varepsilon_1 &= s_{11}\sigma_1 + s_{12}\sigma_2 + s_{13}\sigma_3, & \gamma_{23} &= s_{44}\sigma_{23}, \\ \varepsilon_2 &= s_{12}\sigma_1 + s_{11}\sigma_2 + s_{13}\sigma_3, & \gamma_{31} &= s_{44}\sigma_{31}, \\ \varepsilon_3 &= s_{13}(\sigma_1 + \sigma_2) + s_{33}\sigma_3, & \gamma_{12} &= 2(s_{11} - s_{12})\sigma_{12}, \end{aligned}\right\} \tag{19}$$

where γ_{23}, γ_{31}, and γ_{12} are the engineering shear strains and s_{ij} are the elastic compliances. There are five independent elastic constants, s_{ij}, to be determined, and they are related to the engineering tensile, shear moduli, and Poisson's ratio by

$$\left.\begin{aligned} s_{11} &= \frac{1}{\bar{E}_1}, & s_{33} &= \frac{1}{\bar{E}_3}, \\ s_{13} &= -\frac{\bar{\nu}_{31}}{\bar{E}_3}, & & \\ s_{44} &= \frac{1}{\bar{G}_{23}}, & 2(s_{11} - s_{12}) &= \frac{1}{\bar{G}_{12}}. \end{aligned}\right\} \tag{20}$$

Two categories of fundamental solutions are needed for implementing eqn (4) to determine s_{ij}. The first is for a penny-shaped crack embedded in the principal plane of a transversely isotropic solid subject to remote tension, σ_{33}^0, or shear, σ_{31}^0 and σ_{32}^0 (Fabrikant, 1989). The primary result for the present analysis is the energy release ψ of a penny-shaped crack with a radius a,

$$\psi = \tfrac{1}{2}a^3\{H_t(\sigma_{33}^0)^2 + H_s[(\sigma_{31}^0)^2 + (\sigma_{32}^0)^2]\}, \tag{21}$$

where H_t and H_s are functions of the elastic constants s_{ij} as given in Appendix A. The second is a spherical inclusion embedded in a transversely isotropic matrix subject to general remote stressing (Eshelby, 1957; Mura, 1982). The primary result for the present analysis is the uniform strain $\bar{\varepsilon}_{ij}$ in the inclusion.

$$\left.\begin{array}{ll} \bar{\varepsilon}_1 = L_{11}\varepsilon_1^0+L_{12}\varepsilon_2^0+L_{13}\varepsilon_3^0, & \bar{\gamma}_{23} = L_{44}\gamma_{23}^0, \\ \bar{\varepsilon}_2 = L_{21}\varepsilon_1^0+L_{22}\varepsilon_2^0+L_{23}\varepsilon_3^0, & \bar{\gamma}_{31} = L_{44}\gamma_{31}^0, \\ \bar{\varepsilon}_3 = L_{31}\varepsilon_1^0+L_{32}\varepsilon_2^0+L_{33}\varepsilon_3^0, & \bar{\gamma}_{12} = L_{66}\gamma_{12}^0, \end{array}\right\} \tag{22}$$

where L_{ij} are functions of the elastic constants s_{ij} of composite material and the moduli of inclusions, as given in Appendix B; ε_1^0, ε_2^0 and ε_3^0 are the remote tensile strains; and γ_{23}^0, γ_{31}^0 and γ_{12}^0 are the remote engineering shear strains. These remote strains are related to remote stress σ_{ij}^0 by eqn (19).

The following five sets of remote stressing are applied to eqn (4) to determine the five independent elastic constants, s_{ij}: (1) remote hydrostatic tension, $\sigma_{ij}^0 = \sigma^0\delta_{ij}$; (2) remote uniaxial tension, $\sigma_{33}^0 = \sigma^0$, others $= 0$; (3) remote axisymmetric tension, $\sigma_{11}^0 = \sigma_{22}^0 = \sigma^0$, others $= 0$; (4) remote in-plane shear, $\sigma_{12}^0 = \sigma_{21}^0 = \tau^0$, others $= 0$; and (5) remote out-of-plane shear, $\sigma_{23}^0 = \sigma_{32}^0 = \tau^0$, others $= 0$. These loading sets lead to the following governing equation of s_{ij}:

$$2s_{11}+2s_{12}+4s_{13}+s_{33} = \frac{3-6\nu}{E}+c\left(1-\frac{1-2\nu}{1-2\nu_I}\frac{E_I}{E}\right)\frac{\bar{\varepsilon}_{kk}^{(1)}}{\sigma^0}+H_t\varepsilon, \tag{23a}$$

$$s_{33} = \frac{1}{E}+c\left[\left(1-\frac{1+\nu}{1+\nu_I}\frac{E_I}{E}\right)\frac{\bar{\varepsilon}_{33}^{(2)}}{\sigma^0}+\frac{\nu-\nu_I}{(1-2\nu_I)(1+\nu_I)}\frac{E_I}{E}\frac{\bar{\varepsilon}_{kk}^{(2)}}{\sigma^0}\right]+H_t\varepsilon, \tag{23b}$$

$$s_{11}+s_{12} = \frac{1-\nu}{E}+c\left[\left(1-\frac{1+\nu}{1+\nu_I}\frac{E_I}{E}\right)\frac{\bar{\varepsilon}_{11}^{(3)}}{\sigma^0}+\frac{\nu-\nu_I}{(1-2\nu_I)(1+\nu_I)}\frac{E_I}{E}\frac{\bar{\varepsilon}_{kk}^{(3)}}{\sigma^0}\right], \tag{23c}$$

$$s_{11}-s_{12} = \frac{1+\nu}{E}+c\left(1-\frac{1+\nu}{1+\nu_I}\frac{E_I}{E}\right)L_{66}(s_{11}-s_{12}), \tag{23d}$$

$$s_{44} = \frac{2(1+\nu)}{E}+c\left(1-\frac{1+\nu}{1+\nu_I}\frac{E_I}{E}\right)L_{44}s_{44}+H_s\varepsilon, \tag{23e}$$

where $\varepsilon_{ij}^{(k)}$ are the strains in the inclusion embedded in the composite material subject to the kth set of remote stressing. They are related to the elastic constants, s_{ij}, of composite material by

$$\frac{\bar{\varepsilon}_{kk}^{(1)}}{\sigma^0} = \sum_{i=1}^{3}[(L_{i1}+L_{i2})(s_{11}+s_{12}+s_{13})+L_{i3}(2s_{13}+s_{33})], \tag{24a}$$

$$\frac{\bar{\varepsilon}_{33}^{(2)}}{\sigma^0} = (L_{31}+L_{32})s_{13}+L_{33}s_{33}, \tag{24b}$$

$$\frac{\bar{\varepsilon}_{kk}^{(2)}}{\sigma^0} = \sum_{i=1}^{3}[(L_{i1}+L_{i2})s_{13}+L_{i3}s_{33}], \tag{24c}$$

$$\frac{\bar{\varepsilon}_{11}^{(3)}}{\sigma^0} = (L_{11}+L_{12})(s_{11}+s_{12})+2L_{13}s_{13}, \tag{24d}$$

$$\frac{\bar{\varepsilon}_{kk}^{(3)}}{\sigma^0} = \sum_{i=1}^{3}[(L_{i1}+L_{i2})(s_{11}+s_{12})+2L_{i3}s_{13}]. \tag{24e}$$

In the limiting case, $\varepsilon = 0$ (no cracks), the composite material becomes isotropic. Equations (23) and (24) degrade to the governing equation by Budiansky (1965) for a matrix with spherical inclusions embedded, i.e. eqns (11) and (12) with $\varepsilon = 0$. The other limiting case, $c = 0$ (no inclusions), i.e. parallel penny-shaped cracks in a matrix material, has been studied by Hoenig (1979). He presented the tensile modulus, $\bar{E}_3$, normal to crack planes and shear modulus, $\bar{G}_{23}$, versus the crack density, ε.

The engineering moduli, $\bar{E}_1$, $\bar{E}_3$, $\bar{G}_{12}$ and $\bar{G}_{23}$ of microcracked composites are normalized by the corresponding moduli of the matrix material. Some interesting conclusions are summarized below:

(1) The dependence of all the engineering moduli $\bar{E}_1$, $\bar{E}_3$, $\bar{G}_{12}$ and $\bar{G}_{23}$ on the Poisson's ratios of the matrix and inclusions ν and ν_I is rather weak for the practical range of Poisson's ratios for engineering materials, $0.2 \leqslant \nu,\ \nu_I \leqslant 0.4$. The difference is usually less than 3%. The Poisson's ratios of the matrix and inclusions are fixed as $\nu = 0.25$ and $\nu_I = 0.33$, respectively, in the following.

(2) The in-plane tensile modulus, $\bar{E}_1$, and shear modulus, $\bar{G}_{12}$, which are in the directions parallel to the crack planes, are extremely insensitive to the crack density, ε. For the inclusion volume concentration, c, ranging from 0 to 0.3 and various ratios of matrix/inclusions moduli, E_I/E, the differences of in-plane moduli are always less than 3%. Thus,

$$\bar{E}_1 \cong \bar{E}_1(c, E_I/E) \quad \text{and} \quad \bar{G}_{12} \cong \bar{G}_{12}(c, E_I/E). \tag{25}$$

This means that parallel cracks have little or no effect on the moduli parallel to crack planes and the in-plane tensile and shear moduli, $\bar{E}_1$ and $\bar{G}_{12}$, are very well approximated by Budiansky's (1965) and Hill's (1965) self-consistent estimate for an isotropic composite material, i.e. eqns (11) and (12) with $\varepsilon = 0$.

(3) The normalized out-of-plane tensile modulus, $\bar{E}_3/E$, and shear modulus, $\bar{G}_{23}/G$, are presented versus the crack density, ε, for inclusion volume concentration $c = 0, 0.1, 0.2$ and 0.3 in Figs 4 and 5. The case for $c = 0$ gives Hoenig's (1979) results. The inclusion modulus ratio, E_I/E, is fixed at 1/3 (compliant inclusions). It is observed from the numerical results that the dependence of out-of-plane moduli on inclusion volume concentration, c, and crack density, ε, is approximately decoupled, i.e.

$$\frac{\bar{E}_3}{E} \cong \frac{\bar{E}_3(c, \varepsilon = 0, E_I/E)}{E} \cdot \frac{\bar{E}_3(c = 0, \varepsilon)}{E}, \tag{26a}$$

$$\frac{\bar{G}_{23}}{G} \cong \frac{\bar{G}_{23}(c, \varepsilon = 0, E_I/E)}{G} \cdot \frac{\bar{G}_{23}(c = 0, \varepsilon)}{G}, \tag{26b}$$

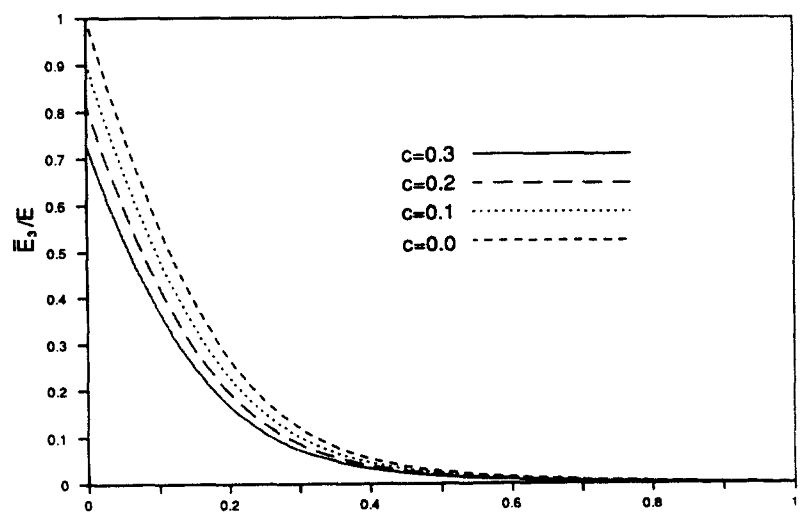

Fig. 4. Variation of $\bar{E}_3/E$ versus crack density ε: Parallel cracks and inclusions $E_I/E = 1/3$.

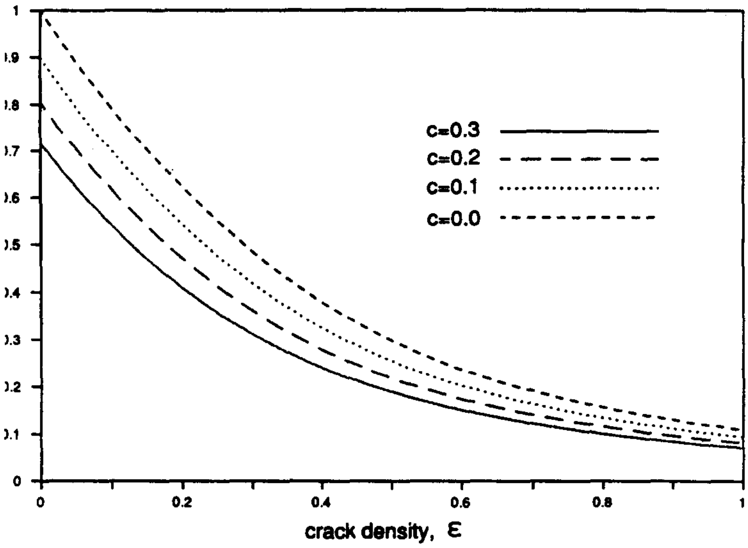

Fig. 5. Variation of $\bar{G}_{23}/G$ versus crack density ε: Parallel cracks and inclusions $E_I/E = 1/3$.

where the first terms on the right-hand side as the normalized moduli of the uncracked composite, i.e. the self-consistent estimate by Budiansky (1965) and Hill (1965) and eqns (11) and (12) with $\varepsilon = 0$; and the second terms on the right-hand side are the normalized moduli of a matrix material with parallel cracks (no inclusions), as shown in Figs 4 and 5 for $c = 0$, i.e. the result obtained by Hoenig (1979). The error to this approximation by eqn (26) is less than 8% for $c \leqslant 0.3$. One can estimate the moduli of a cracked composite material by the moduli of the uncracked composite and the modulus reduction curves by Hoenig, i.e. the curves $c = 0$ in Figs 4 and 5.

(4) The engineering Poisson's ratio, $\bar{v}_{31}$, depends rather weakly on the inclusion volume concentration, c. A good approximation to $\bar{v}_{31}$ is $\bar{v}_{31}$ $(c = 0, \varepsilon, v)$ for a microcracked matrix material without inclusions,

$$\bar{v}_{31} \cong \bar{v}_{31}(c = 0, \varepsilon, v) = v\frac{\bar{E}_3(c = 0, \varepsilon, v)}{E}, \tag{27}$$

where $\bar{E}_3(c = 0, \varepsilon, v)$ is the tensile modulus of the cracked matrix, i.e. the curve $c = 0$ in Fig. 4. Its dependence on v is rather weak.

Based on the discussions above, one finds that the moduli of a composite with parallel cracks are completely determined by the moduli of the uncracked composite and Hoenig's (1979) modulus reduction curve.

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APPENDIX A

The fundamental solution for a penny-shaped crack embedded in the principal plane of a transversely isotropic solid subject to remote tension or shear stress can be found in the book by Fabrikant (1989). The following coefficients, H_t and H_s, in the expression of energy release ψ [eqn (21)] are given as

$$H_t = \tfrac{8}{3}(\gamma_1+\gamma_2)\left(s_{33}-\frac{s_{13}^2}{s_{11}}\right),$$

$$H_s = \frac{16}{3}\,\frac{\sqrt{2(s_{11}-s_{12})s_{44}\gamma_1\gamma_2(\gamma_1+\gamma_2)(s_{11}s_{33}-s_{13}^2)}}{\sqrt{2(s_{11}-s_{12})s_{44}s_{11}+\gamma_1\gamma_2(\gamma_1+\gamma_2)(s_{11}s_{33}-s_{13}^2)}},$$

where s_{ij} are the elastic constants of the transversely isotropic solid, γ_1 and γ_2 are the roots of the following fourth-order polynomial of γ with the positive imaginary part ($\mathrm{Im}\,\gamma_1 > 0$, $\mathrm{Im}\,\gamma_2 > 0$):

$$(s_{11}s_{33}-s_{13}^2)\gamma^4-[2s_{13}(s_{11}-s_{12})+s_{11}s_{44}]\gamma^2+s_{11}^2-s_{12}^2 = 0.$$

APPENDIX B

The strains $\bar{\varepsilon}_{ij}$ in a spherical inclusion embedded in a general anisotropic matrix subject to remote stressing σ_{ij}^0 (or remote strains ε_{ij}^0) are uniform (Eshelby, 1957). If the matrix is transversely isotropic, an explicit expression between inclusion strains $\bar{\varepsilon}_{ij}$ and remote strains ε_{ij}^0 [eqn (22)] can be found (Mura, 1982). The coefficients L_{ij} relating $\bar{\varepsilon}_{ij}$ to ε_{ij}^0 in eqn (22) are given by

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} = \left\{\mathbf{I}+\begin{pmatrix} S_{1111} & S_{1122} & S_{1133} \\ S_{1122} & S_{1111} & S_{1133} \\ S_{3311} & S_{3311} & S_{3333} \end{pmatrix}\right.$$

$$\left.\times\left[\frac{E_I}{(1+\nu_I)(1-2\nu_I)}\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{11} & s_{13} \\ s_{13} & s_{13} & s_{33} \end{pmatrix}\begin{pmatrix} 1-\nu_I & \nu_I & \nu_I \\ \nu_I & 1-\nu_I & \nu_I \\ \nu_I & \nu_I & 1-\nu_I \end{pmatrix}-\mathbf{I}\right]\right\}^{-1},$$

$$L_{44} = \{1+2S_{1313}[s_{44}G_I-1]\}^{-1}, \quad L_{66} = \{1+2S_{1212}[2(s_{11}-s_{12})G_I-1]\}^{-1},$$

where $\mathbf{I}$ is the identity matrix; E_I, ν_I and G_I are the Young's modulus, Poisson's ratio, and shear modulus of inclusions, respectively; s_{ij} are the elastic constants of the matrix; and S_{ijkl} are the so-called Eshelby tensors, which can be found in Mura (1982) for a transversely isotropic solid.